Vector Space Concepts

ECE 275A – Statistical Parameter Estimation

Ken Kreutz-Delgado
ECE Department, UC San Diego

Heuristic Concept of a Linear Vector Space

Many important physical, engineering, biological, sociological, economic, scientific quantities, which we call **vectors**, have the following conceptual properties.

- There exists a natural or conventional 'zero point' or "origin", the **zero vector, 0,**.
- Vectors can be **added** in a symmetric **commutative** and **associative** manner to produce other vectors

$$z = x + y = y + x\,, \quad x,y,z \text{ are vectors} \qquad \text{(commutativity)}$$

$$x + y + z \triangleq x + (y + z) = (x + y) + z\,, \quad x,y,z \text{ are vectors} \qquad \text{(associativity)}$$

- Vectors can be scaled by the (symmetric) multiplication of scalars (**scalar multiplication**) to produce other vectors

$$z = \alpha x = x\alpha\,, \quad x,z \text{ are vectors, } \alpha \text{ is a scalar} \qquad \text{(scalar multiplication of } x \text{ by}$$

  - The scalars can be members of any fixed **field** (such as the field of rational polynomials). We will work only with the fields of real and complex numbers.
- Each vector x has an **additive inverse,** $\mathbf{-x = (-1)x}$

$$x - x \triangleq x + (-1)x = x + (-x) = 0$$

Formal Concept of a Linear Vector Space

- A **Vector Space**, $\mathcal{X}$, is a set of **vectors**, $x \in \mathcal{X}$, over a **field**, $\mathcal{F}$, of **scalars**.
  - If the scalars are the field of real numbers, then we have a **Real** Vector Space.
  - If the scalars are the field of complex numbers, then we have a **Complex** Vector Space.
- Any vector $x \in \mathcal{X}$ can be multiplied by an arbitrary scalar α to form $\alpha x = x \alpha \in \mathcal{X}$. This is called **scalar multiplication.**
  - Note that we must have **closure of scalar multiplication**. I.e, we demand that the new vector formed via scalar multiplication **must also be** in $\mathcal{X}$.
- Any two vectors $x, y \in \mathcal{X}$ can be added to form $x + y \in \mathcal{X}$ where the operation "$+$" of **vector addition** is associative and commutative.
  - Note that we must have **closure of vector addition**.
- The vector space $\mathcal{X}$ must contain an **additive identity** (the **zero vector** $\mathbf{0}$) and, for every vector x, an **additive inverse** $-x$.
- In this course we primarily consider **finite dimensional** vector spaces $\dim \mathcal{X} = n < \infty$ and mostly give results appropriate for this restriction.

Linear Vector Spaces – Cont.

- Any vector x in an n-dimensional vector space can be represented (with respect to an appropriate basis–see below) as an n-tuple ($n \times 1$ column vector) over the field of scalars,

$$x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathcal{X} = \mathcal{F}^n = \mathbb{C}^n \text{ or } \mathbb{R}^n .$$

  - We refer to this as a **canonical representation** of a finite-dimensional vector. We often (but not always) assume that vectors in an n-dimensional vector space are canonically represented by $n \times 1$ column vectors.

Linear Vector Spaces – Cont.

- Any **linear combination** of arbitrarily selected vectors $x_1, \cdots, x_r$ drawn from the space $\mathcal{X}$
$$\alpha_1 x_1 + \cdots + \alpha_r x_r$$
for arbitrary r, and scalars α_i, $i = 1, \cdots, r$, must also be a vector in $\mathcal{X}$.
  - This is easily shown via induction using the properties of closure under pairwise vector addition, closure under scalar multiplication, and associativity of vector addition.
  - This **global** 'closure of linear combinations property of $\mathcal{X}$' (i.e., the property holds **everywhere** on $\mathcal{X}$) is why we often refer to $\mathcal{X}$ as a (globally) **Linear** Vector Space.
  - This is in contradistinction to **locally** linear spaces, such as differentiable manifolds, of which the surface of a ball is the classic example of a space which is locally linear (flat) but globally curved.
- Some important physical phenomenon of interest **cannot** be modeled by linear vector spaces, the classic example being rotations of a rigid body in three dimensional space (this is because finite (i.e., non-infinitesimal) rotations do not commute.)

Examples of Vectors

Voltages, Currents, Power, Energy, Forces, Displacements, Velocities, Accelerations, Temperature, Torques, Angular Velocities, Income, Profits, ... , can all be modeled as vectors.

Example: Set of all $m \times n$ matrices. Define matrix addition by component-wise addition and scalar multiplication by component-wise multiplication of the matrix component by the scalar. This is easily shown to be a vector space.

- We can place the elements of this mn-dimensional vector space into *canonical form* by stacking the columns of an $m \times n$ matrix A to form an $mn \times 1$ column vector denoted by $\mathrm{vec}(A)$ (sometimes also denoted by $\mathrm{stack}(A)$).

Example: Take

$$\mathcal{X} = \{f(t) = x_1 \cos(\omega_1 t) + x_2 \cos(\omega_2 t) \text{ for } -\infty < t < \infty; x_1, x_2 \in \mathbb{R}; \omega_1 \neq \omega_2\}$$

and define vector addition and scalar multiplication component wise. Note that any vector $f \in \mathcal{X}$ has a *canonical representation* $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. Thus $\mathcal{X} \cong \mathcal{X}' \triangleq \mathbb{R}^2$, and without loss of generality (wlog) we often work with $\mathcal{X}'$ in lieu of $\mathcal{X}$.

Examples of Vectors – Cont.

Important Example: Set of all Functions forms a Vector Space

- Consider functions (say of time t) f and g, which we sometimes also denote as $f(\cdot)$ and $g(\cdot)$.
  - $f(t)$ is the **value** of the function f at time t. (Think of $f(t)$ as a sample of f taken at time t.) Strictly speaking, then, $f(t)$ is **not** the function f itself.
- Functions are **single-valued** **by definition**. Therefore

$$f(t) = g(t)\,,\ \forall t \quad \Longleftrightarrow \quad f = g$$

  I.e., functions are **uniquely** defined once we know their output values **for all** possible input values t
- We can define vector addition to create a new function $h = f + g$ by specifying the value of $h(t)$ for all t, which we do as follows:

$$h(t) = (f+g)(t) \triangleq f(t) + g(t)\,,\ \forall t$$

- We define scalar multiplication of the function f by the scalar α to create a new function $g = (\alpha f)$ via

$$(\alpha f)(t) = \alpha \cdot f(t)\,,\ \forall t$$

- Finally we define the zero function o as the function that maps to the scalar value 0 for all t, $o(t) = 0\,,\ \forall t$.

Vector Subspaces

- A subset $\mathcal{V} \subset \mathcal{X}$ is a **subspace** of a vector space $\mathcal{X}$ if it is a vector space in its own right.
- If $\mathcal{V}$ is a subspace of a vector space $\mathcal{X}$, we call $\mathcal{X}$ the **parent space** or **ambient space** of $\mathcal{V}$.
  - **It is understood that a subspace $\boldsymbol{\mathcal{V}}$ "inherits" the vector addition and scalar multiplication operations from the ambient space $\boldsymbol{\mathcal{X}}$.** To be a subspace, $\mathcal{V}$ must also inherit the zero vector element.
    - Given this fact, to determine if a subset $\mathcal{V}$ is also a subspace one needs to check that every linear combination of vectors in $\mathcal{V}$ yields a vector in $\mathcal{V}$.
    - This latter property is called the property of **closure of the subspace $\boldsymbol{\mathcal{V}}$ under linear combinations of vectors in $\boldsymbol{\mathcal{V}}$**.
      Therefore if closure fails to hold for a subset $\mathcal{V}$, then $\mathcal{V}$ is **not** a vector subspace.

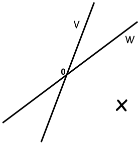
      Note that testing for closure includes as a special case testing whether the zero vector belongs to $\mathcal{V}$.

Vector Subspaces - Cont.

Consider the complex vector space $\mathcal{X}$ = *complex* $n \times n$ matrices, $n > 1$, with matrix addition and scalar multiplication defined component-wise. Are the following subsets of $\mathcal{X}$ vector subspaces?

- $\mathcal{V}$ = upper triangular matrices. This is a subspace as it is closed under the operations of scalar multiplication and vector addition inherited from $\mathcal{X}$.
- $\mathcal{V}$ = positive definite matrices. This is not a subspace as it is not closed under scalar multiplication. (Or, even simpler, it does not contain the zero element.)
- $\mathcal{V}$ = symmetric matrices, $A = A^T$. This is a subspace as it is closed under the operators inherited from $\mathcal{X}$.
- $\mathcal{V}$ = hermitian matrices, $A = A^H$ (the set of complex symmetric matrices where $A^H = \overline{(A^T)} = (\bar{A})^T$). This is not a subspace as it is not closed under scalar multiplication (check this!). It *does* include the zero element.

Subspace Sums



- Given two **subsets** $\mathcal{V}$ and $\mathcal{W}$ of vectors, we define their **set sum** by

$$\mathcal{V} + \mathcal{W} = \{ v + w \,|\, v \in \mathcal{V} \text{ and } w \in \mathcal{W} \} \ .$$

- Let the sets $\mathcal{V}$ and $\mathcal{W}$ in addition both be **subspaces** of $\mathcal{X}$. In this case we call $\mathcal{V} + \mathcal{W}$ a **subspace sum** and we have
  - $\mathcal{V} \cap \mathcal{W}$ and $\mathcal{V} + \mathcal{W}$ are also subspaces of $\mathcal{X}$
  - $\mathcal{V} \cup \mathcal{W} \subset \mathcal{V} + \mathcal{W}$ where in general $\mathcal{V} \cup \mathcal{W}$ is **not** a subspace.
- In general, we have the following ordering of subspaces,

$$0 \triangleq \{0\} \subset \mathcal{V} \cap \mathcal{W} \subset \mathcal{V} + \mathcal{W} \subset \mathcal{X} \, ,$$

where $\{0\}$ is the **trivial subspace** consisting only of the zero vector (additive identity) of $\mathcal{X}$. The trivial subspace has dimension zero.

Linear Independence

- By definition r vectors $x_1, \cdots, x_r \in \mathcal{X}$ are **linearly independent** when,

$$\alpha_1 x_1 + \cdots + \alpha_r x_r = 0 \quad \text{if and only if} \quad \alpha_1 = \cdots = \alpha_r = 0$$

- Suppose this condition is violated because (say) $\alpha_1 \neq 0$, then we have

$$x_1 = -\frac{1}{\alpha_1} (\alpha_2 x_2 + \cdots + \alpha_r x_r)$$

- A collection of vectors are **linearly dependent** if they are **not** linearly independent.

Linear Independence - Cont.

- Assume that the r vectors, x_i, are canonically represented, $x_i \in \mathcal{F}^n$.
- Then the definition of linear independence can be written in matrix-vector form as

$$X\alpha = \begin{pmatrix} x_1 & \cdots & x_r \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = 0 \quad \Longleftrightarrow \quad \alpha \triangleq \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = 0$$

- Thus $x_1, \cdots, x_r$ are linearly independent iff the associated $n \times r$ matrix

$$X \triangleq \begin{pmatrix} x_1 & \cdots & x_r \end{pmatrix}$$

has full column rank (equivalently, iff the null space of X is trivial).

Span of a Set of Vectors

- The **span** of the collection $x_1, \cdots, x_r \in \mathcal{X}$ is the set of all linear combinations of the vectors,

$$\text{Span}\,\{x_1, \cdots, x_r\} = \{y \mid y = \alpha_1 x_1 + \cdots + \alpha_r x_r = X\alpha,\ \forall\, \alpha \in \mathcal{F}^r\} \subset \mathcal{X}$$

- The set $\mathcal{V} = \text{Span}\,\{x_1, \cdots, x_r\}$ is a vector subspace of $\mathcal{X}$.
- If, in addition, the spanning vectors $x_1, \cdots, x_r$ are linearly independent we say that the collection is a **linearly independent spanning set** or a **basis** for the subspace $\mathcal{V}$.
- We denote a basis for a subspace $\mathcal{V}$ by

$$B_{\mathcal{V}} = \{x_1, \cdots, x_r\}$$

Basis and Dimension

- Given a basis for a vector space or subspace, the **number of basis vectors in the basis is unique.**

- For a given space or subspace, there are many different bases, but they must **all have the same number of vectors.**

- This number, then, is **an intrinsic property of the space itself** and is called the **dimension** $\boldsymbol{d = \dim \mathcal{V}}$ of the space or subspace $\mathcal{V}$.

  If the number of elements, d, in a basis is finite, we say that the space is **finite dimensional,** otherwise we say that the space is **infinite dimensional.**

- **Linear algebra** is the study of linear mappings between *finite* dimensional vector spaces. The study of linear mappings between **infinite** dimensional vector spaces is known as **Linear Functional Analysis** or **Linear Operator Theory.**

Basis and Dimension – Cont.

- The dimension of the trivial subspace is zero, $0 = \dim\{0\}$.
- If $\mathcal{V}$ is a subspace of $\mathcal{X}$, $\mathcal{V} \subset \mathcal{X}$, we have $\dim \mathcal{V} \leq \dim \mathcal{X}$.
- In general for two arbitrary subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{X}$ we have,

$$\dim (\mathcal{V} + \mathcal{W}) = \dim \mathcal{V} + \dim \mathcal{W} - \dim (\mathcal{V} \cap \mathcal{W}) \ ,$$

  and

$$0 \leq \dim (\mathcal{V} \cap \mathcal{W}) \leq \dim (\mathcal{V} + \mathcal{W}) \leq \dim \mathcal{X} \ .$$

- Furthermore, if $\mathcal{X} = \mathcal{V} + \mathcal{W}$ then,

$$\dim \mathcal{X} \leq \dim \mathcal{V} + \dim \mathcal{W} \, ,$$

  with equality if and only if $\mathcal{V} \cap \mathcal{W} = \{0\}$.

Independent Subspaces and Projections

- Two subspaces, $\mathcal{V}$ and $\mathcal{W}$, of a vector space $\mathcal{X}$ are **independent** or **disjoint** when $\mathcal{V} \cap \mathcal{W} = \{0\}$. In this case we have

$$\dim(\mathcal{V} + \mathcal{W}) = \dim \mathcal{V} + \dim \mathcal{W} .$$

- If $\mathcal{X} = \mathcal{V} + \mathcal{W}$ for two *independent* subspaces $\mathcal{V}$ and $\mathcal{W}$ we say that $\mathcal{V}$ and $\mathcal{W}$ are **companion subspaces** and we write,

$$\mathcal{X} = \mathcal{V} \oplus \mathcal{W} .$$

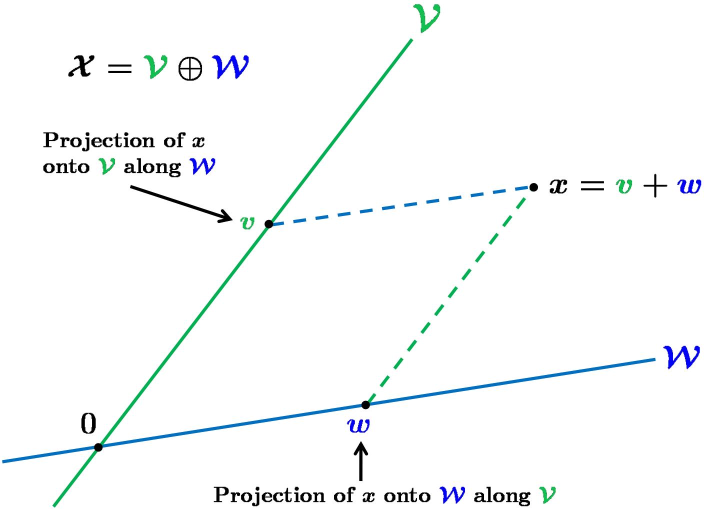
  In this case $\dim \mathcal{X} = \dim \mathcal{V} + \dim \mathcal{W}$.

  Given two companion subspaces $\mathcal{V}$ and $\mathcal{W}$ any vector $x \in \mathcal{X}$ can be written **uniquely** as

$$x = v + w , \quad v \in \mathcal{V} \text{ and } w \in \mathcal{W} .$$

  - The unique component $\boldsymbol{v}$ is called **the projection of x onto $\mathcal{V}$ along its companion space $\mathcal{W}$.**
  - The unique component $\boldsymbol{w}$ is called **the projection of x onto $\mathcal{W}$ along its companion space $\mathcal{V}$.**

Independent Subspaces and Projections – Cont.



Projection Operators

- Given the unique decomposition of a vector x along two companion subspaces $\mathcal{V}$ and $\mathcal{W}$, $x = v + w$, we define the **companion projection operators** $P_{\mathcal{V}|\mathcal{W}}$ and $P_{\mathcal{W}|\mathcal{V}}$ by,

$$P_{\mathcal{V}|\mathcal{W}}\, x \triangleq v \quad \text{and} \quad P_{\mathcal{W}|\mathcal{V}}\, x = w$$

- Obviously $P_{\mathcal{V}|\mathcal{W}} + P_{\mathcal{W}|\mathcal{V}} = I$. I.e., $P_{\mathcal{V}|\mathcal{W}} = I - P_{\mathcal{W}|\mathcal{V}}$.
- It is straightforward to show that $P_{\mathcal{V}|\mathcal{W}}$ and $P_{\mathcal{W}|\mathcal{V}}$ are both **idempotent,**

$$P^2_{\mathcal{V}|\mathcal{W}} = P_{\mathcal{V}|\mathcal{W}} \quad \text{and} \quad P^2_{\mathcal{W}|\mathcal{V}} = P_{\mathcal{W}|\mathcal{V}}$$

  where $P^2_{\mathcal{V}|\mathcal{W}} = \left(P_{\mathcal{V}|\mathcal{W}}\right)\,\left(P_{\mathcal{V}|\mathcal{W}}\right)$. For example

$$P^2_{\mathcal{V}|\mathcal{W}}\, x = P_{\mathcal{V}|\mathcal{W}}\,\left(P_{\mathcal{V}|\mathcal{W}}\, x\right) = P_{\mathcal{V}|\mathcal{W}}\, v = v = P_{\mathcal{V}|\mathcal{W}}\, x$$

  and since this is true **for all** $x \in \mathcal{X}$ it must be the case that $P^2_{\mathcal{V}|\mathcal{W}} = P_{\mathcal{V}|\mathcal{W}}$.
- It can also be shown that the projection operators $P_{\mathcal{V}|\mathcal{W}}$ and $P_{\mathcal{W}|\mathcal{V}}$ are **linear operators**.

Linear Operators and Matrices

Consider a function A which maps between two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, $A : \mathcal{X} \to \mathcal{Y}$.

- $\mathcal{X}$ is called the **input space** or the **source space** or the **domain.**
- $\mathcal{Y}$ is called the **output space** or the **target space** or the **codomain.**
- The **mapping** or **operator** A is said to be **linear** if

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2 \quad \forall x_1, x_2 \in \mathcal{X},\ \forall \alpha_1, \alpha_2 \in \mathcal{F}\,.$$

  - Note that in order for this definition to be well-posed the vector spaces $\mathcal{X}$ and $\mathcal{Y}$ both must have the same field of scalars $\mathcal{F}$.
    - For example, $\mathcal{X}$ and $\mathcal{Y}$ must be both real vectors spaces, or must be both complex vector spaces.

Linear Operators and Matrices - Cont.

- It is well-known that any linear operator between finite dimensional vectors spaces has a matrix representation.
  - In particular if $n = \dim \mathcal{X} < \infty$ and $m = \dim \mathcal{Y} < \infty$ for two vector spaces over the field $\mathcal{F}$, then a linear operator A which maps between these two spaces has an $m \times n$ matrix representation over the field $\mathcal{F}$.
  - Note that projection operators on finite-dimensional vector spaces must have matrix representations as a *consequence* of their linearity.
  - Often, for convenience, we assume that any such linear mapping A *is* an $m \times n$ matrix and we write $A \in \mathcal{F}^{m\times n}$.
- **Example:** Differentiation as a linear mapping between 2nd order polynomials

$$b + 2c\,x = \frac{d}{dx}\left(a + b\,x + c\,x^2\right) \quad \Longleftrightarrow \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

using the simple polynomial basis functions 1, x, and x^2. If a different set of polynomial basis functions are used, then we would have a different vector-matrix representation of the differentiation. **Again we note: representations of vectors and operators are basis dependent.**

Two Linear Operator Induced Subspaces

- Every linear operator has two natural vector subspaces associated with it.

  The **Range Space** (or **Image**),

  $$\mathcal{R}(A) \triangleq A(\mathcal{X}) \triangleq \{y \mid y = Ax, \; x \in \mathcal{X}\} \subset \mathcal{Y} \,,$$

  The **Nullspace** (or **Kernel**),

  $$\mathcal{N}(A) = \{x \mid Ax = 0\} \in \mathcal{X} \,.$$

  - Note that the nullspace is a subspace of the source space (domain), while the range space is a subspace of the target space (the codomain).
  - It is straightforward to show that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are linear subspaces using the fact that A is a **linear** operator.

- When attempting to solve a linear problem $y = Ax$, a solution exists if and only if $y \in \mathcal{R}(A)$.

  - If $y \in \mathcal{R}(A)$ we say that the problem is **consistent.** Otherwise the problem is **inconsistent.**

Two Linear Operator Induced Subspaces – Cont.

- The dimension of the range space of a linear operator A is the **rank** of A,

$$r(A) = \text{rank}(A) = \dim \mathcal{R}(A)\,,$$

- The dimension of the nullspace of a linear operator A is the **nullity** of A,

$$\nu(A) = \text{nullity}(A) = \dim \mathcal{N}(A)\,,$$

- The rank and nullity of a linear operator A have unique values which are independent of the specific matrix representation of A. They are **intrinsic** properties of the linear operator A and **invariant** with respect to all changes of representation. Note that, as dimensions, the rank and nullity must take on nonnegative integer or zero values.
  - Given a matrix representation for $A \in \mathcal{F}^{m\times n}$, standard undergraduate courses in linear algebra explain how to determine the rank and nullity via LU factorization (aka Gaussian elimination) to place a matrix into upper echelon form. The rank, $r = r(A)$ is then given by the number of nonzero pivots while the nullity, $\nu = \nu(A)$, is given by $\nu = n - r$.

Linear Forward and Inverse Problem

- Given a linear mapping between two vector spaces $A : \mathcal{X} \to \mathcal{Y}$ the problem of computing an "output" y in the codomain given an "input" vector x in the domain,

  $$Ax \underset{\to}{=} y$$

  is called the **forward problem.**

  - **The forward problem is typically well-posed** in that knowing A and given x one can construct y by (say) a straightforward matrix-vector multiplication.

- Given a vector y in the codomain, the problem of determining an x in the domain for which

  $$y \underset{\to}{=} Ax$$

  is known as an **inverse problem.**

  - Solving the linear inverse problem is much harder than solving the forward problem, even when the problem is well-posed.

    Furthermore the **inverse problem is often ill-posed** compounding the problem difficulty

Well-Posed and Ill-Posed Linear Inverse Problems

Given am m-dimensional vector y in the codomain, the **inverse problem** of determining an n-dimensional vector x in the domain for which $Ax = y$ is said to be **well-posed** if and only if the following three conditions are true for the linear mapping A:

1. $y \in \mathcal{R}(A)$ **for all** $y \in \mathcal{Y}$ **so that a solution exists for all** y**.** I.e., we demand that A be **onto,** $\mathcal{R}(A) = \mathcal{Y}$ or, equivalently, that $r(A) = m$. It it not enough to merely require consistency for a given y because even the tiniest error or misspecification in y can render the problem inconsistent.
2. If a solution exists, we demand that it be unique. I.e., we demand that that A be **one-to-one,** $\mathcal{N}(A) = \{0\}$. Equivalently, $\nu(A) = 0$.
3. The solution x does not depend sensitively on the value of y. I.e., we demand that A be **numerically well-conditioned.**

If **any** of these three conditions is violated we say that the inverse problem is **ill-posed.**

Condition three is studied in great depth in courses on Numerical Linear Algebra. In this course, we ignore the numerical conditioning problem and focus on the first two conditions only.